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Instructive Properties of Quantized Gravitating Dust Shell

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Abstract

We investigate quantum dynamics of self-gravitating spherical dust shell. The wave functions of discrete spectrum are not localized inside the Schwarzschild radius. We argue that such shells can transform into white holes (in another space). It is plausible that shells with bare masses larger than the Planck mass loose their mass emitting lighter shells.

Thin dust shell is one of the simplest models of collapsing gravitating bodies. Equations of motion of such objects were derived in ref. [1]. The classical dynamics of this system was considered in refs. [2, 3, 4]. This model was quantized in various nonequivalent ways with physically different results in papers [5, 6, 7, 8]. In our opinion, the most natural approach was proposed in ref. [7] where the problem was reduced to the usual s-wave Klein-Gordon equation in a Coulomb field. In this note we would like to point out some curious effects arising in this approach which may turn instructive for more realistic situations.

Let us briefly present the basic steps leading to the Klein-Gordon equation for this system. The situation is rather unusual because there are two classical equations

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of motion for the shell radius r [1]:

$$\frac{\ddot{r} + km/r^2}{\sqrt{1 + \dot{r}^2 - 2km/r}} + \frac{\ddot{r}}{\sqrt{1 + \dot{r}^2}} = 0; \tag{1}$$

$$\frac{\ddot{r} + km/r^2}{\sqrt{1 + \dot{r}^2 - 2km/r}} - \frac{\ddot{r}}{\sqrt{1 + \dot{r}^2}} = \frac{k\mu}{r^2}. \tag{2}$$

These two equations for a single variable are consistent, as will be demonstrated below, since they depend on the total mass m (or the total energy) of the shell as defined by the Schwarzschild metric outside the shell. It means in fact that the equations depend on the initial conditions. In these equations μ is the "bare" mass of the shell, i.e. the rest mass of each particle of the dust times their number, k is the Newton gravitational constant, and $\dot{r} = dr/d\tau$ where τ is the proper time of the dust. First integrals of these equations can be written respectively as

$$\sqrt{1 + \dot{r}^2 - 2km/r} \; + \; \sqrt{1 + \dot{r}^2} \; = C_1; \tag{3}$$

$$\sqrt{1 + \dot{r}^2 - 2km/r} \; - \; \sqrt{1 + \dot{r}^2} \; = C_2 - \frac{k\mu}{r}. \tag{4}$$

Multiplying these two expressions we obtain:

$$\frac{2km}{r} = C_1 \left(\frac{k\mu}{r} - C_2 \right) \tag{5}$$

¿From the last equality it follows that the integrals (3) and (3) are compatible only if $C_2 = 0$ and $C_1 = 2m/\mu$. If these conditions are imposed, the equations of motion (1) and (2) are consistent and equivalent to

$$\frac{\ddot{r}}{\sqrt{1 + \dot{r}^2}} = -\frac{k\mu}{2r^2}. \tag{6}$$

Equation (6) has the following first integral

$$m = \mu\sqrt{1 + \dot{r}^2} - \frac{k\mu^2}{2r}. \tag{7}$$

This expression will be taken as a classical Hamiltonian of the system (after going over from the velocity $\dot{r}$ to the canonical momentum p). ¿From the formal point of view (of the theory of differential equations, or analytical mechanics) any function of a first integral can be chosen as a Hamiltonian. The choice is by no means unique, but eq. (7) is singled out because this Hamiltonian corresponds to the total energy of the shell, and this energy m has an explicit and simple expression.

One can easily recognize in the rhs of eq. (7) the energy of a relativistic particle in a Coulomb-like field, $-k\mu^2/2r$, written in proper time τ. It is convenient to go over from τ to the world time t inside the shell:

$$d\tau = dt\sqrt{1-v^2},\ v = dr/dt.$$

We obtain now:

$$m = \frac{\mu}{\sqrt{1-v^2}} - \frac{k\mu^2}{2r}. \tag{8}$$

Clearly in this case the canonical momentum equals $p = \mu v/\sqrt{1-v^2}$ and the Hamiltonian has the well known form:

$$H = \sqrt{p^2+\mu^2} - \frac{k\mu^2}{2r}. \tag{9}$$

The quantum-mechanical wave equation corresponding to this Hamiltonian is derived by the standard procedure, taking the square of the root, i.e., rewriting (9) as:

$$(H + k\mu^2/2r)^2 = p^2 + \mu^2.$$

Thus one obtains the usual Klein-Gordon radial equation for s-wave:

$$\left(\partial_r^2 + \frac{2}{r}\,\partial_r + \frac{k\mu^2 m}{r} + \frac{k\mu^2}{4r^2} + m^2 - \mu^2\right)\psi = 0. \tag{10}$$

The discrete spectrum of this equation is well-known [9, 10]:

$$m_n = \mu\left[1 + \frac{k^2\mu^4}{\left(2n+1+\sqrt{1-k^2\mu^4}\right)^2}\right]^{-1/2} \tag{11}$$

The radial quantum number n is integer and runs from 0 to infinity.

The spectrum has a singularity at $k\mu^2 = 1$. At larger values of μ the r^{-2} potential becomes so strong that the "fall to the center" takes place, i.e. there are no stationary states. It looks natural that for heavy pressureless matter (with $\mu > m_{Pl} = 1/\sqrt{k}$) naive quantum mechanical effects cannot stop the collapse.

The curious property of the states belonging to the discrete spectrum should be pointed out. Even in the most tightly bound ground state for $k\mu^2 = 1$ the wave function is not localized inside the gravitational radius of the shell. The probability to find the shell outside it is $3/e^2 \approx 0.4$. Here we naively use r as the operator of coordinate. In the relativistic case a more refined definition of the coordinate should be used [11, 7]. It is clear however that any reasonable definition of the coordinate operator would not change the localization considerably.

Let us consider now the continuous spectrum. Though we assume as above that $k\mu^2 < 1$ or in other words $\mu < m_{Pl}$, the total energy m can be arbitrarily large. The wave function of such a state is a superposition of incoming and outgoing spherical waves with equal amplitudes. It is evidently non-localized. To consider the quantum analogue of the collapse of the classical shell we have to turn to wave packets. As was noted in ref. [7] the gravitational radius of this object will be as smeared as its energy. Let us assume that the initial radius of the maximum of the incoming spherical wave packet is much larger than its average gravitational radius r_g. ¿From the point of view of a distant observer this packet moving towards the center freezes at $r = r_g$. However, in its proper time it reaches the center in a finite interval $\delta\tau$, bounces back and then after the same time interval returns to its initial position and form. Certainly, it returns not to "our" space, but to a quite different one. This is a possible realization of the white hole phenomenon [12, 13].

The considered realization of a white hole based on quantum scattering differs essentially from the classical examples of white holes. In the classical case the very

existence of the phenomenon depends crucially upon the presence of singularity at $r = 0$, while in our case the transformation of an incoming spherical wave into an outgoing one takes place even for nonsingular potentials.

Finally let us return to the case of a large bare mass, $\mu > m_{Pl}$. This problem formally coincides with that of a charged scalar particle in the field of a point-like nucleus with a supercritical electric charge, $Z\alpha > 1$. It is known that the vacuum around a supercharged nucleus is unstable and this nucleus discharges by emitting positively charged particles (see e.g. refs. [14, 15]). A similar scenario is plausible for our problem: the collapsing shell loses its bare mass μ by emitting light shell-lets till it reaches the subcritical mass. This phenomenon would resemble quantum evaporation of usual black holes. On the other hand, in the subcritical situation, $\mu < m_{Pl}$, the emission of shell-lets does not take place (even if the physical mass m is larger than m_{Pl}). It can be considered as a hint that small black holes do not evaporate, though at $m > \mu$ they form white holes in another universe.

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